Derivation of Equations of Motion for a Four Link Robotic Leg for Walking Vehicle

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Abstract

A four degree of freedom leg for a walking robot has been modeled using Newton's method. Unlike robot manipulators, which have a fixed base, a leg model must include inertial forces due to base motion. These forces have been included in the formulation. These equations can be used for design, simulation, and control. The inverse kinematics for this leg are also presented. This allows the joint angles to be computed from a desired foot-hold position.

Introduction

Walking robots have been a topic of research and imagination since antiquity (Raibert, 1986). In the nineteenth century, mechanisms to achieve a repetitious gait were developed. These 'walking horses' suffered from the disadvantage that they could not automatically compensate for uneven terrain. Developments in automatic control theory and electronics have generated a resurgence in research into walking vehicles.

Applications of walking vehicles include interplanetary or off-road exploration, nuclear power-plant clean-up, or transportation for the handicapped (a walking "wheel" chair). These applications require a vehicle which can propel a payload while decoupling that payload from uneven terrain.

Although a walking vehicle has many advantages over a wheeled vehicle, it suffers from technical disadvantages. Since the vehicles legs, or active suspension, have many degrees of freedom, design, construction, and control are more difficult and expensive tasks. Studies at JPL[1] have indicated that wheeled vehicles are more energy efficient than legged vehicles, a key disadvantage when energy resources are limited. Legs must also be light-weight and strong, since they must carry their own weight as well as the vehicle's payload (Eltze, et al, 1995).

Initial control strategies focussed on quasi-static approaches (Klein, et al, 1983). This involves updating the control signals to the legs so that a subset of the legs forms a static, stable platform (Klein, et al, 1987; Liu, et al, 1997). The drawbacks to this strategy include intensive inverse kinematic calculations and slow vehicle speed. This limitation comes partly from the force distribution problem, which requires a quasi-static formulation to avoid foot force discontinuity through the transition between ground-contacting legs (Gardner, 1991).

Modern control strategies eliminate some of these drawbacks. The dynamically stable controller (Raibert, 1990) converts the set of legs into an equivalent single leg which dynamically balances the center of mass of the vehicle. This vehicle is always falling in the right direction to achieve the desired motion. Several single- and multi-leg vehicles which use this strategy have been developed and demonstrated successfully.

New developments involve biologically inspired control strategies (Berns, et al, 1999). These strategies use a paradigm derived from the nervous system of cockroaches or cats to generate nonlinear coupled oscillators which generate the control signals. This approach is simple to implement. It suffers the drawback of predictability. The controller is adaptive and requires some heuristic refinement to perform properly.

Another control approach is to use state-space based adaptive or nonlinear controllers. One example uses a model reference adaptive controller (Lee, et al, 1986). State space controllers

1. Private communication with Dr. Eddie Tunstel, Robotics Vehicles Group, Jet Propulsion Laboratory.

need model information governing how the actuators interact with the system they are controlling. In the case of a walking robot, this involves modeling the leg dynamics. A similar task occurs in the development of manipulator control systems (Asada, et al 1986). However, the manipulator base is fixed, and the dynamics of the body do not influence the control or modeling of the manipulator. The controller presented in Lee, et al, 1986 does not include body motion in the model of the leg accelerations. This presents a severe drawback in the system's performance.

Regardless of the control strategy employed, it is desirable to test the controller in simulation prior to building and testing hardware. Consequently, equations of motion for a new configuration, including body dynamics, must be derived and simulated.

The work presented in this paper involve a new leg configuration. The equations of motion are derived using Newton's method. The inverse kinematics for this leg configuration are also presented.

Derivation of the Leg Equations of Motion

The basic design for this project uses four-link, bottom-mounted legs, similar to insect legs, where the fourth link is a flexible foot containing both a restoring spring and a force sensor system (see Fig. 1). This design has three controlled degrees of freedom, which allow the foot to be positioned arbitrarily within the limits of the link lengths. The foot-spring stores energy during foot placement and releases it when the foot leaves contact with the ground. This compliance is similar to the ankle in most mammals and has been used in shoe design to increase walking efficiency.

The development of the EOM is a time-consuming, error-prone task (Asada, et al 1986), especially when the six DOF for body motion are included. Newton's method is used to determine the equations of motion. Although this method requires knowledge and experience to apply it, it is more efficient than Lagrange's method for this complicated case.

A Free-Body-Diagram (FBD) of each link and of the foot is shown in Fig. 2. The foot is subject to ground forces, $\underset{\sim}{F}_G$, gravity, $\underset{\sim}{G}_4$, and a constraint force, $\underset{\sim}{F}_4$, and torque, $\underset{\sim}{\tau}_4$, exerted by the preceding link.On each link other than the foot, the forces acting are the negative of the constraint exerted by the following link, $-\underset{\sim}{F}_{i+1}$ and $-\underset{\sim}{\tau}_{i+1}$, gravity, $\underset{\sim}{G}_i$, and the forces exerted by the preceding link, $\underset{\sim}{F}_i$ and $\underset{\sim}{\tau}_i$. The inertial terms are the rate of change of linear momentum for the link, $\frac{d(\underset{\sim}{p}_i)}{dt} = \frac{d(m_i\underset{\sim}{v}_i)}{dt} = m_i\frac{d\underset{\sim}{v}_i}{dt}$, and the rate of change of angular momentum about the link's center of mass, $\frac{d(\underset{\sim}{I}_i \bullet \underset{\sim}{\omega}_i)}{dt}$. Here, $\underset{\sim}{v}_i$ is the velocity of the link's center of mass, m_i is the link's mass, $\underset{\sim}{\omega}_i$ is the link's angular velocity, and $\underset{\sim}{I}_i$ is the link's moment of inertia tensor.

A vector sum of the forces and moments about the center of mass acting on each FBD is performed and generates four sets of two vector equations.

$$\underset{\sim}{F}_4 + \underset{\sim}{F}_G + \underset{\sim}{G}_4 = m_4\frac{d\underset{\sim}{v}_4}{dt} \tag{1}$$

$$\underset{\sim}{\tau}_4 - \underset{\sim}{x}_{C4} \times \underset{\sim}{F}_4 + (\underset{\sim}{x}_4 - \underset{\sim}{x}_{C4}) \times \underset{\sim}{F}_G = \frac{d(\underset{\sim}{I}_4 \bullet \underset{\sim}{\omega}_4)}{dt} \tag{2}$$

$$\underset{\sim}{F}_i - \underset{\sim}{F}_{i+1} + \underset{\sim}{G}_i = m_i\frac{d\underset{\sim}{v}_i}{dt} \quad i = 1,2,3 \tag{3}$$

$$\underset{\sim}{\tau}_i - \underset{\sim}{\tau}_{i+1} - \underset{\sim}{x}_{Ci} \times \underset{\sim}{F}_i - (\underset{\sim}{x}_i - \underset{\sim}{x}_{Ci}) \times \underset{\sim}{F}_{i+1} = \frac{d(\underset{\sim}{I}_i \bullet \underset{\sim}{\omega}_i)}{dt} \quad i = 1,2,3 \tag{4}$$

In Equations 1 through 4, $\underset{\sim}{x}_{Ci}$ is the location of the center of mass of link i with respect to joint i-1, and $\underset{\sim}{x}_i$ is the location of joint i with respect to joint i-1.

Equations 1 through 4 contain constraint forces, $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$, $\underset{\sim}{F}_3$, $\underset{\sim}{F}_4$, which must be eliminated. Later, if these forces are required for design work, they can be explicitly determined. When the four vector contraint forces are eliminated, the eight vector equations become four vector equations, containing only the constraint torques, gravity forces, and inertial terms.

$$\underset{\sim}{\tau}_4 = \frac{d(\underset{\sim}{I}_4 \bullet \underset{\sim}{\omega}_4)}{dt} + \underset{\sim}{x}_{C4} \times \left[m_4 \frac{d\underset{\sim}{v}_4}{dt} - \underset{\sim}{G}_4 \right] - \underset{\sim}{x}_4 \times \underset{\sim}{F}_G \quad (5)$$

$$\begin{aligned} \underset{\sim}{\tau}_3 = {} & \frac{d(\underset{\sim}{I}_3 \bullet \underset{\sim}{\omega}_3 + \underset{\sim}{I}_4 \bullet \underset{\sim}{\omega}_4)}{dt} + \underset{\sim}{x}_{C3} \times \left[m_3 \frac{d\underset{\sim}{v}_3}{dt} - \underset{\sim}{G}_3 \right] + (\underset{\sim}{x}_3 + \underset{\sim}{x}_{C4}) \times \left[m_4 \frac{d\underset{\sim}{v}_4}{dt} - \underset{\sim}{G}_4 \right] \\ & -(\underset{\sim}{x}_3 + \underset{\sim}{x}_4) \times \underset{\sim}{F}_G \end{aligned} \quad (6)$$

$$\begin{aligned} \underset{\sim}{\tau}_2 = {} & \frac{d(\underset{\sim}{I}_2 \bullet \underset{\sim}{\omega}_2 + \underset{\sim}{I}_3 \bullet \underset{\sim}{\omega}_3 + \underset{\sim}{I}_4 \bullet \underset{\sim}{\omega}_4)}{dt} + \underset{\sim}{x}_{C2} \times \left[m_2 \frac{d\underset{\sim}{v}_2}{dt} - \underset{\sim}{G}_2 \right] + (\underset{\sim}{x}_2 + \underset{\sim}{x}_{C3}) \times \left[m_3 \frac{d\underset{\sim}{v}_3}{dt} - \underset{\sim}{G}_3 \right] + \\ & (\underset{\sim}{x}_2 + \underset{\sim}{x}_3 + \underset{\sim}{x}_{C4}) \times \left[m_4 \frac{d\underset{\sim}{v}_4}{dt} - \underset{\sim}{G}_4 \right] - (\underset{\sim}{x}_2 + \underset{\sim}{x}_3 + \underset{\sim}{x}_4) \times \underset{\sim}{F}_G \end{aligned} \quad (7)$$

$$\begin{aligned} \underset{\sim}{\tau}_1 = {} & \frac{d(\underset{\sim}{I}_1 \bullet \underset{\sim}{\omega}_1 + \underset{\sim}{I}_2 \bullet \underset{\sim}{\omega}_2 + \underset{\sim}{I}_3 \bullet \underset{\sim}{\omega}_3 + \underset{\sim}{I}_4 \bullet \underset{\sim}{\omega}_4)}{dt} + \underset{\sim}{x}_{C1} \times \left[m_1 \frac{d\underset{\sim}{v}_1}{dt} - \underset{\sim}{G}_1 \right] + \\ & (\underset{\sim}{x}_1 + \underset{\sim}{x}_{C2}) \times \left[m_2 \frac{d\underset{\sim}{v}_2}{dt} - \underset{\sim}{G}_2 \right] + (\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_{C3}) \times \left[m_3 \frac{d\underset{\sim}{v}_3}{dt} - \underset{\sim}{G}_3 \right] + \\ & (\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3 + \underset{\sim}{x}_{C4}) \times \left[m_4 \frac{d\underset{\sim}{v}_4}{dt} - \underset{\sim}{G}_4 \right] - (\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3 + \underset{\sim}{x}_4) \times \underset{\sim}{F}_G \end{aligned} \quad (8)$$

So far, these equations are general. Any leg composed of four separate links will follow these equations. The details of a particular configuration depend on the evaluation of the time derivatives.

To proceed ruther, the accelerations of the centers of mass (CM) are required. The positions of the link CMs are (Fig. 3):

$$\underset{\sim}{R}_{Ci} = \underset{\sim}{R}_0 + \underset{\sim}{X}_0 + \sum_{j=1}^{i-1} \underset{\sim}{x}_j + \underset{\sim}{x}_{Ci}, \text{ i=1,2,3,4,} \tag{9}$$

where $\underset{\sim}{R}_0$ is the location of the vehicle's center of mass and $\underset{\sim}{X}_0$ is the location of the shoulder joint with respect to the vehicle's center of mass.

In order to evaluate derivatives, it is necessary to define coordinate systems in which to express the joint and CM positions. The first, inertial coordinate system is fixed to the ground at some convenient reference point. The unit vectors for this system are $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$, where $\underset{\sim}{E}_1$ is initially aligned with the vehicle's direction of travel, $\underset{\sim}{E}_2$ is aligned with gravity, and $\underset{\sim}{E}_3$ is orthogonal to both $\underset{\sim}{E}_1$ and $\underset{\sim}{E}_2$ in a right hand sense. This coordinate system will be thrown away once velocities are evaluated.

The second coordinate system is affixed to the vehicle body's CM and has unit vectors, $(\underset{\sim}{e}_{01}, \underset{\sim}{e}_{02}, \underset{\sim}{e}_{03})$, which are initially aligned with $(\underset{\sim}{E}_1, \underset{\sim}{E}_2, \underset{\sim}{E}_3)$. The lower case e's for the unit vectors indicate that this system is rotating and not inertial. The coordinate transformation between the inertial system and the body fixed coordinate system is expressed in terms of Euler angles (Greenwood, 1965):

$$[A]_{0I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

where the angular velocity for this transformation is:

$$\underset{\sim}{\omega}_0 = (\dot{\phi} - \dot{\psi}\sin\theta)\underset{\sim}{e}_{01} + (\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta)\underset{\sim}{e}_{02} + (\dot{\psi}\cos\phi\cos\theta - \dot{\theta}\sin\phi)\underset{\sim}{e}_{03} \tag{12}$$

The next coordinate system, $(\underset{\sim}{e}_{11}, \underset{\sim}{e}_{12}, \underset{\sim}{e}_{13})$, is located at the shoulder joint and rotates relative to the body with angle, q_1 (see Fig. 4). The coordinate transformation between the $(\underset{\sim}{e}_{01}, \underset{\sim}{e}_{02}, \underset{\sim}{e}_{03})$ system and the $(\underset{\sim}{e}_{11}, \underset{\sim}{e}_{12}, \underset{\sim}{e}_{13})$ system is:

$$[A]_{10} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin q_1 & \cos q_1 & 0 \\ -\cos q_1 & -\sin q_1 & 0 \end{bmatrix}. \tag{13}$$

The angular velocity of the $(\underset{\sim}{e}_{11}, \underset{\sim}{e}_{12}, \underset{\sim}{e}_{13})$ system with respect to the $(\underset{\sim}{e}_{01}, \underset{\sim}{e}_{02}, \underset{\sim}{e}_{03})$ system is:

$$\underset{\sim}{\Omega}_1 = \dot{q}_1\underset{\sim}{e}_{11}. \tag{14}$$

The next three coordinate systems are similarly defined (see Fig. 5). The unit vector, $\underset{\sim}{e}_{i1}$, points from the i-1th joint to the ith joint through the center of mass. The unit vector, $\underset{\sim}{e}_{i3}$, is aligned with the motor torque. The final unit vector, $\underset{\sim}{e}_{i2}$, is orthogonal to $\underset{\sim}{e}_{i1}$ and $\underset{\sim}{e}_{i3}$. The joint angle, q_i, is the angle between $\underset{\sim}{e}_{i1}$ and $\underset{\sim}{e}_{i+1,1}$.

The coordinate transformation tensor for each of these systems is:

$$[A]_{i,i-1} = \begin{bmatrix} \cos q_i & \sin q_i & 0 \\ -\sin q_i & \cos q_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad i=2,3,4 \tag{15}$$

and the angular velocity of system i with respect to system i-1 is:

$$\underset{\sim}{\Omega}_i = \dot{q}_i \underset{\sim}{e}_{i3} \quad i=2,3,4. \tag{16}$$

Since each coordinate system is chained to the previous one, angular velocities of the joint-based reference frames are:

$$\underset{\sim}{\omega}_i = \underset{\sim}{\omega}_{i-1} + \underset{\sim}{\Omega}_i, \; i=1,2,3,4. \tag{17}$$

The accelerations for use in equations 5 through 8 are determined by taking two derivatives of equation 9. This is a tedious process which results in the accelerations as a function of the Euler angles and the joint angles.

Only four of the twelve equations 5 through 8 contain information which is useful. The other eight contain information about the eight constraint torques in the pin joints. The torque provided by the motor (or torsion spring) at the joint is the component in the $\underset{\sim}{e}_{i3}$ direction. Otherwise the joint is free to move in that direction. The torques, $\underset{\sim}{\tau}_1 = \tau_1 \underset{\sim}{e}_{11}$ and $\underset{\sim}{\tau}_i = \tau_i \underset{\sim}{e}_{i3}$ for i=2,3,4, are the independent variables in these equations. The joint accelerations are the dependent variables.

The EOM for this leg configuration are:

$$[H]\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\\ \ddot{q}_3\\ \ddot{q}_4\end{bmatrix} + [B]\begin{bmatrix}\dot{q}_1^2\\ \dot{q}_1\dot{q}_2\\ \dot{q}_1\dot{q}_3\\ \dot{q}_1\dot{q}_4\\ \dot{q}_2^2\\ \dot{q}_2\dot{q}_3\\ \dot{q}_3^2\end{bmatrix} + g[G]\begin{bmatrix}\cos q_1 \sin\theta - \sin q_1 \sin\phi \cos\theta\\ \cos\phi\cos\theta\\ \sin q_1 \sin\theta + \cos q_1 \sin\phi\cos\theta\end{bmatrix} = [A]\begin{bmatrix}F_{G1}\\ F_{G2}\\ F_{G3}\end{bmatrix} + \begin{bmatrix}d_1\\ d_2\\ d_3\\ d_4\end{bmatrix} + \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\\ -kq_4\end{bmatrix}. \tag{18}$$

The matrices, [H], [B], [G], and [A] can be written by defining the following constants:

$$\begin{array}{cc} S_2 = \sin(q_2) & C_2 = \cos(q_2) \\ S_{23} = \sin(q_2 + q_3) & C_{23} = \cos(q_2 + q_3) \\ S_{234} = \sin(q_2 + q_3 + q_4) & C_{234} = \cos(q_2 + q_3 + q_4) \\ \alpha_{11} = x_{C2}S_2 & \alpha_{21} = x_{C2}C_2 \\ \alpha_{12} = x_{C3}S_{23} & \alpha_{22} = x_{C3}C_{23} \\ \alpha_{13} = x_{C4}S_{234} & \alpha_{23} = x_{C4}C_{234} \\ \beta_{11} = L_2S_2 & \beta_{21} = L_2C_2 \\ \beta_{12} = L_3S_{23} & \beta_{22} = L_3C_{23} \\ \beta_{13} = L_4S_{234} & \beta_{23} = L_4C_{234} \\ \gamma_{11} = L_3S_4 & \gamma_{21} = L_3C_4 \\ \gamma_{12} = L_2S_{34} & \gamma_{22} = L_2C_{34} \\ \gamma_{13} = L_2S_3 & \gamma_{23} = L_2C_3 \\ \gamma_{14} = L_4 & \gamma_{24} = L_3C_{34} \end{array} \tag{19}$$

$$\begin{array}{c} J_1 = I_{4,33} + m_4x_{C4}^2 \\ J_2 = J_1 + I_{3,33} + m_3x_{C3}^2 \\ J_3 = J_2 + I_{2,33} + m_2x_{C2}^2 \\ K_1 = 2(I_{2,11} - I_{2,22})S_2C_2 \\ K_2 = 2(I_{3,11} - I_{3,22})S_{23}C_{23} \\ K_3 = 2(I_{4,11} - I_{4,22})S_{234}C_{234} \end{array} \tag{20}$$

The mass matrix, [H], is:

$$[H] = \begin{bmatrix} h_{11} & 0 & 0 & 0 \\ 0 & h_{22} & h_{23} & h_{24} \\ 0 & h_{23} & h_{33} & h_{34} \\ 0 & h_{24} & h_{34} & h_{44} \end{bmatrix}, \tag{21}$$

where $h_{11} = I_{1,11} + I_{2,11}C_2^2 + I_{2,22}S_2^2 + m_2\alpha_{11}^2 + I_{3,11}C_{23}^2 + I_{3,22}S_{23}^2 + m_3(\beta_{11} + \alpha_{12})^2 + I_{4,11}C_{234}^2 + I_{4,22}S_{234}^2 + m_4(\beta_{11} + \beta_{12} + \alpha_{13})^2$,

$h_{22} = J_3 + m_3(L_2^2 + 2x_{C3}\gamma_{23}) + m_4[L_2^2 + L_3^2 + 2(L_3\gamma_{23} + x_{C4}(\gamma_{24} + \gamma_{21}))]$,

$h_{23} = h_{32} = J_2 + m_4L_3^2 + (m_3x_{C3} + m_4L_3)\gamma_{23} + m_4x_{C4}(2\gamma_{21} + \gamma_{22})$, $h_{33} = J_2 + m_4(L_3^2 + 2x_{C4}\gamma_{21})$, $h_{42} = h_{24} = J_1 + m_4x_{C4}(\gamma_{22} + \gamma_{21})$,

$h_{43} = h_{34} = J_1 + m_4x_{C4}\gamma_{21}$, and $h_{44} = I_{4,33} + m_4x_{C4}^2$.

The damping matrix, [B], is:

$$[B] = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 & 0 & b_{26} & b_{27} \\ b_{31} & 0 & 0 & 0 & b_{35} & 0 & 0 \\ b_{41} & 0 & 0 & 0 & b_{45} & b_{46} & b_{47} \end{bmatrix}, \quad (22)$$

where $b_{12} = K_1 + K_2 + K_3 + 2m_2\alpha_{11}\alpha_{21} + 2m_3(\beta_{11} + \alpha_{12})(\beta_{21} + \alpha_{22}) + 2m_4(\beta_{11} + \beta_{12} + \alpha_{13})(\beta_{21} + \beta_{22} + \alpha_{23})$,

$b_{13} = K_2 + K_3 + 2m_3(\beta_{11} + \alpha_{12})\alpha_{22} + 2m_4(\beta_{11} + \beta_{12} + \alpha_{13})(\beta_{22} + \alpha_{23})$, $b_{14} = K_3 + 2m_4(\beta_{11} + \beta_{12} + \alpha_{13})\alpha_{23}$,

$b_{21} = \frac{1}{2}(K_1 + K_2 + K_3) - m_2\alpha_{11}\alpha_{21} - m_3(\beta_{11} + \alpha_{12})(\beta_{21} + \alpha_{22}) - m_4(\beta_{11} + \beta_{12} + \alpha_{13})(\beta_{21} + \beta_{22} + \alpha_{23})$, $b_{26} = -2(m_3x_{C3}\gamma_{13} + m_4x_{C4}\gamma_{12})$,

$b_{27} = -(m_3x_{C3}\gamma_{13} + m_4x_{C4}\gamma_{12})$, $b_{31} = \frac{1}{2}(K_2 + K_3) - m_3(\beta_{11} + \alpha_{12})\alpha_{22} - m_4(\beta_{11} + \beta_{12} + \alpha_{13})(\beta_{22} + \alpha_{23})$,

$b_{35} = (m_3x_{C3} + m_4L_3)\gamma_{13} + m_4x_{C4}\gamma_{12}$, $b_{41} = \frac{1}{2}K_3 - m_4(\beta_{11} + \beta_{12} + \alpha_{13})\alpha_{23}$, $b_{45} = m_4x_{C4}(\gamma_{11} + \gamma_{12} + x_{C4})$,

$b_{46} = 2m_4x_{C4}(\gamma_{11} + \gamma_{12} + x_{C4})$, $b_{47.} = m_4x_{C4}(\gamma_{11} + \gamma_{12} + x_{C4})$.

The force transmission matrix, [A], is:

$$[A] = \begin{bmatrix} 0 & 0 & a_1 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & 0 \\ 0 & a_6 & 0 \end{bmatrix}, \quad (23)$$

where $a_1 = -\beta_{11} - \beta_{12} - \beta_{13}$, $a_2 = \gamma_{11} + \gamma_{12}$, $a_3 = \gamma_{21} + \gamma_{22} + \gamma_{14}$, $a_4 = \gamma_{11}$, $a_5 = \gamma_{21} + \gamma_{14}$, and $a_6 = \gamma_{14}$.

The gravity matrix, [G], is:

$$[G] = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_1 & g_2 \\ 0 & g_3 & g_4 \\ 0 & g_5 & g_6 \end{bmatrix}, \tag{24}$$

where $g_1 = m_2\alpha_{11} + m_3(\beta_{11} + \alpha_{12}) + m_4(\beta_{11} + \beta_{12} + \alpha_{13})$,

$g_2 = -m_2\alpha_{21} - m_3(\beta_{21} + \alpha_{22}) - m_4(\beta_{21} + \beta_{22} + \alpha_{23})$, $g_3 = m_3\alpha_{12} + m_4(\beta_{12} + \alpha_{13})$, $g_4 = -m_3\alpha_{22} - m_4(\beta_{22} + \alpha_{23})$, $g_5 = m_4\alpha_{13}$,

and $g_6 = -m_4\alpha_{23}$.

In order to evaluate the inertial term, the following terms are defined:

$$\tilde{\alpha}_0 = \frac{d\tilde{\omega}_0}{dt} \tag{25}$$

$$\tilde{a}_1 = \frac{d^2\tilde{R}_0}{dt^2} + \tilde{\alpha}_0 \times \tilde{X}_0 + \tilde{\omega}_0 \times (\tilde{\omega}_0 \times \tilde{X}_0) \tag{26}$$

$$\tilde{p}_1 = \tilde{a}_1 + \tilde{\alpha}_0 \times \tilde{x}_{C1} + \tilde{\omega}_0 \times (\tilde{\omega}_0 \times \tilde{x}_{C1}) + 2\tilde{\omega}_0 \times (\tilde{\Omega}_1 \times \tilde{x}_{C1}) \tag{27}$$

$$\begin{aligned} \tilde{p}_2 = \tilde{a}_1 + \tilde{\alpha}_0 \times (\tilde{L}_1 + \tilde{x}_{C2}) + \tilde{\omega}_0 \times (\tilde{\omega}_0 \times (\tilde{L}_1 + \tilde{x}_{C2})) + \\ 2\tilde{\omega}_0 \times (\tilde{\Omega}_1 \times (\tilde{L}_1 + \tilde{x}_{C2}) + \tilde{\Omega}_2 \times \tilde{x}_{C2}) \end{aligned} \tag{28}$$

$$\begin{aligned} \tilde{p}_3 = \tilde{a}_1 + \tilde{\alpha}_0 \times (\tilde{L}_1 + \tilde{L}_2 + \tilde{x}_{C3}) + \tilde{\omega}_0 \times (\tilde{\omega}_0 \times (\tilde{L}_1 + \tilde{L}_2 + \tilde{x}_{C3})) + \\ 2\tilde{\omega}_0 \times (\tilde{\Omega}_1 \times (\tilde{L}_1 + \tilde{L}_2 + \tilde{x}_{C3}) + \tilde{\Omega}_2 \times (\tilde{L}_2 + \tilde{x}_{C3}) + \tilde{\Omega}_3 \times \tilde{x}_{C3}) \end{aligned} \tag{29}$$

$$\begin{aligned}\underset{\sim}{p}_4 \;=\; &\underset{\sim}{a}_1 + \underset{\sim}{\alpha}_0 \times (\underset{\sim}{L}_1 + \underset{\sim}{L}_2 + \underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) + \underset{\sim}{\omega}_0 \times (\underset{\sim}{\omega}_0 \times (\underset{\sim}{L}_1 + \underset{\sim}{L}_2 + \underset{\sim}{L}_3 + \underset{\sim}{x}_{C4})) + \\ &2\underset{\sim}{\omega}_0 \times (\underset{\sim}{\Omega}_1 \times (\underset{\sim}{L}_1 + \underset{\sim}{L}_2 + \underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) + \underset{\sim}{\Omega}_2 \times (\underset{\sim}{L}_2 + \underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) + \underset{\sim}{\Omega}_3 \times (\underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) + \underset{\sim}{\Omega}_4 \times \underset{\sim}{x}_{C4})\end{aligned} \tag{30}$$

$$\begin{aligned}\underset{\sim}{Q}_1 \;=\; &\underset{\sim}{I}_1 \bullet (\underset{\sim}{\alpha}_0 + \underset{\sim}{\omega}_0 \times \underset{\sim}{\Omega}_1) + [(\underset{\sim}{\omega}_0 \times \underset{\sim}{I}_1) - (\underset{\sim}{I}_1 \times \underset{\sim}{\omega}_0)] \bullet (\underset{\sim}{\omega}_0 + \underset{\sim}{\Omega}_1) + \\ &[(\underset{\sim}{\Omega}_1 \times \underset{\sim}{I}_1) - (\underset{\sim}{I}_1 \times \underset{\sim}{\Omega}_1)] \bullet \underset{\sim}{\omega}_0\end{aligned} \tag{31}$$

$$\begin{aligned}\underset{\sim}{Q}_2 \;=\; &\underset{\sim}{I}_2 \bullet (\underset{\sim}{\alpha}_0 + \underset{\sim}{\omega}_0 \times (\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2)) + [(\underset{\sim}{\omega}_0 \times \underset{\sim}{I}_2) - (\underset{\sim}{I}_2 \times \underset{\sim}{\omega}_0)] \bullet (\underset{\sim}{\omega}_0 + \underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2) + \\ &[((\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2) \times \underset{\sim}{I}_2) - (\underset{\sim}{I}_2 \times (\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2))] \bullet \underset{\sim}{\omega}_0\end{aligned} \tag{32}$$

$$\begin{aligned}\underset{\sim}{Q}_3 \;=\; &\underset{\sim}{I}_3 \bullet (\underset{\sim}{\alpha}_0 + \underset{\sim}{\omega}_0 \times (\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3)) + [(\underset{\sim}{\omega}_0 \times \underset{\sim}{I}_3) - (\underset{\sim}{I}_3 \times \underset{\sim}{\omega}_0)] \bullet (\underset{\sim}{\omega}_0 + \underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3) + \\ &[((\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3) \times \underset{\sim}{I}_3) - (\underset{\sim}{I}_3 \times (\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3))] \bullet \underset{\sim}{\omega}_0\end{aligned} \tag{33}$$

$$\begin{aligned}\underset{\sim}{Q}_4 \;=\; &\underset{\sim}{I}_4 \bullet (\underset{\sim}{\alpha}_0 + \underset{\sim}{\omega}_0 \times (\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3 + \underset{\sim}{\Omega}_4)) + \\ &[(\underset{\sim}{\omega}_0 \times \underset{\sim}{I}_4) - (\underset{\sim}{I}_4 \times \underset{\sim}{\omega}_0)] \bullet (\underset{\sim}{\omega}_0 + \underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3 + \underset{\sim}{\Omega}_4) + \\ &[((\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3 + \underset{\sim}{\Omega}_4) \times \underset{\sim}{I}_3) - (\underset{\sim}{I}_3 \times (\underset{\sim}{\Omega}_1 + \underset{\sim}{\Omega}_2 + \underset{\sim}{\Omega}_3 + \underset{\sim}{\Omega}_4))] \bullet \underset{\sim}{\omega}_0\end{aligned} \tag{34}$$

The vector disturbance terms are:

$$\begin{aligned}\underset{\sim}{d}_1 \;=\; &-\underset{\sim}{Q}_1 - \underset{\sim}{Q}_2 - \underset{\sim}{Q}_3 - \underset{\sim}{Q}_4 - m_1\underset{\sim}{x}_{C1} \times \underset{\sim}{p}_1 - m_2(\underset{\sim}{L}_1 + \underset{\sim}{x}_{C2}) \times \underset{\sim}{p}_2 \\ &-m_3(\underset{\sim}{L}_1 + \underset{\sim}{L}_2 + \underset{\sim}{x}_{C3}) \times \underset{\sim}{p}_3 - m_4(\underset{\sim}{L}_1 + \underset{\sim}{L}_2 + \underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) \times \underset{\sim}{p}_4\end{aligned} \tag{35}$$

$$\underset{\sim}{d}_2 \;=\; -\underset{\sim}{Q}_2 - \underset{\sim}{Q}_3 - \underset{\sim}{Q}_4 - m_2\underset{\sim}{x}_{C2} \times \underset{\sim}{p}_2 - m_3(\underset{\sim}{L}_2 + \underset{\sim}{x}_{C3}) \times \underset{\sim}{p}_3 - m_4(\underset{\sim}{L}_2 + \underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) \times \underset{\sim}{p}_4 \tag{36}$$

$$\underset{\sim}{d}_3 \;=\; -\underset{\sim}{Q}_3 - \underset{\sim}{Q}_4 - m_3\underset{\sim}{x}_{C3} \times \underset{\sim}{p}_3 - m_4(\underset{\sim}{L}_3 + \underset{\sim}{x}_{C4}) \times \underset{\sim}{p}_4 \tag{37}$$

$$\underset{\sim}{d}_4 \;=\; -\underset{\sim}{Q}_4 - m_4\underset{\sim}{x}_{C4} \times \underset{\sim}{p}_4 \tag{38}$$

Only the component from each vector which is aligned with the joint motor torque is required. The resulting vector of inertial torques is:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{d}_1 \bullet \underset{\sim}{e}_{11} \\ \underset{\sim}{d}_2 \bullet \underset{\sim}{e}_{23} \\ \underset{\sim}{d}_3 \bullet \underset{\sim}{e}_{33} \\ \underset{\sim}{d}_4 \bullet \underset{\sim}{e}_{43} \end{bmatrix}. \tag{39}$$

Inverse Kinematics of Leg

Given the desired foot position, $\underset{\sim}{R}_d = R_{d1}\underset{\sim}{E}_1 + R_{d2}\underset{\sim}{E}_2 + R_{d3}\underset{\sim}{E}_3$, the inverse kinematic problem involves determining what joint angles, q_1, q_2, q_3, q_4, generate that position. This can be expressed mathematically as (see Fig. 6):

$$\underset{\sim}{x}_1 + \underset{\sim}{x}_2 + \underset{\sim}{x}_3 + \underset{\sim}{x}_4 = \underset{\sim}{R}_d - \underset{\sim}{R}_0 - \underset{\sim}{X}_0 = \underset{\sim}{X}_d. \tag{40}$$

It is sufficient to specify $\underset{\sim}{X}_d$, the desired foot position relative to the shoulder position.The foot-hold equation becomes:

$$L_1\underset{\sim}{e}_{11} + L_2\underset{\sim}{e}_{21} + L_3\underset{\sim}{e}_{31} + L_4\underset{\sim}{e}_{41} = X_{d1}\underset{\sim}{e}_{01} + X_{d2}\underset{\sim}{e}_{02} + X_{d3}\underset{\sim}{e}_{03} \tag{41}$$

This equation can be written entirely in the $(\underset{\sim}{e}_{01}, \underset{\sim}{e}_{02}, \underset{\sim}{e}_{03})$ coordinate system:

$$[A]_{10}^T\left(\begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} + [A]_{21}^T\begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} + [A]_{21}^T[A]_{32}^T\begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} + [A]_{21}^T[A]_{32}^T[A]_{43}^T\begin{bmatrix} L_4 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} X_{d1} \\ X_{d2} \\ X_{d3} \end{bmatrix}. \tag{42}$$

Since the leg is a four degree of freedom system, one more foot position quantity can be specified. This is the global angle of the foot with respect to the ground, n_{d3}. In order to determine the desired joint angles that give a foot position in global cartesian coordinates, the kinematics of the manipulator must be inverted, yielding:

$$\begin{aligned}
\tan(q_1) &= -\left(\frac{X_{d1}}{X_{d2}}\right) \\
\cos(q_1 + q_2 + q_3) &= n_{d3} \\
\rho_1 &= X_{d3} - L_1 - L_4\cos(q_2 + q_3 + q_4) \\
\rho_2 &= \cos(q_1)X_{d1} - \sin(q_1)X_{d2} - L_4\sin(q_2 + q_3 + q_4) . \\
\cos(q_3) &= \frac{\rho_1^2 + \rho_2^2 - L_2^2 - L_3^2}{2L_2L_3} \\
\tan(q_2) &= \frac{(L_3\cos(q_3) + L_2)\rho_2 - L_3\sin(q_3)\rho_1}{(L_3\cos(q_3) + L_2)\rho_1 + L_3\sin(q_3)\rho_2}
\end{aligned} \tag{43}$$

Conclusions

Equations of motion for a four link manipulator have been derived. These equations include terms resulting from base motion. The equations can be used for simulation to test controllers, for state space controller formulations, and for optimization of the leg parameters.

Inverse kinematics for this leg configuration have been derived. These equations can be used to calculate joint angles to achieve desired foot position.

Future Work

Further work needs to be done in the basic leg design and the total vehicle design. In particular, the correct optimization criteria need to be chosen and the significant independent variables identified.

Acknowledgments

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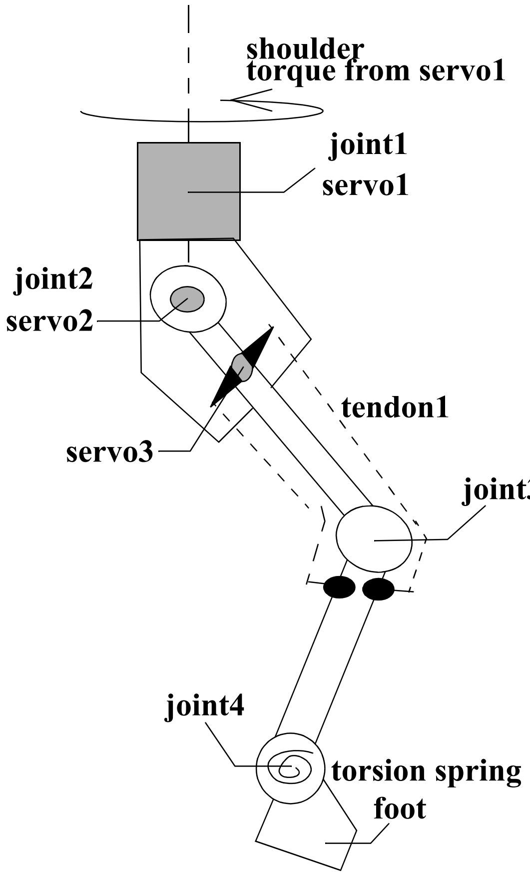

Figure 1. Tendon Based Articulated Leg

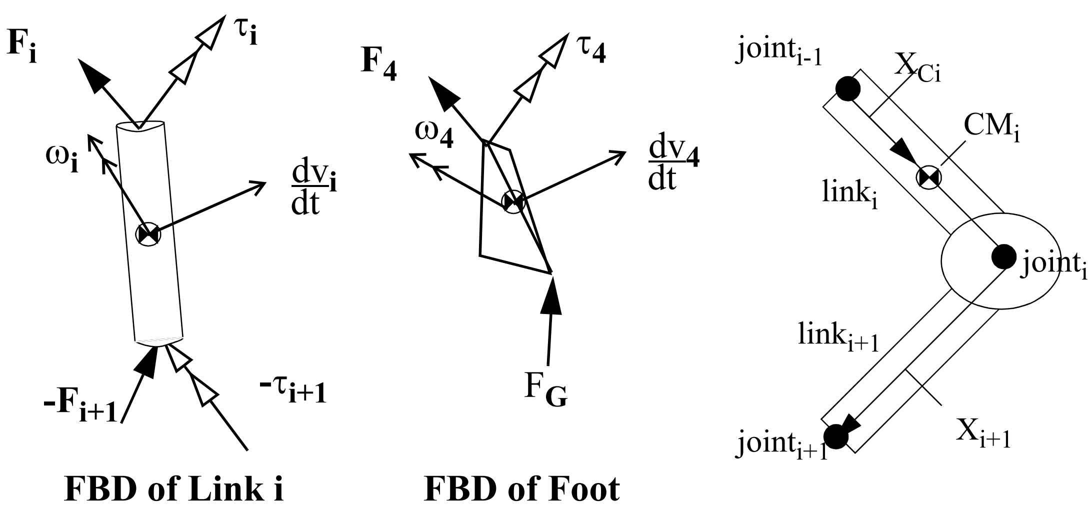

Figure 2. Free Body Diagram of Link and Foot

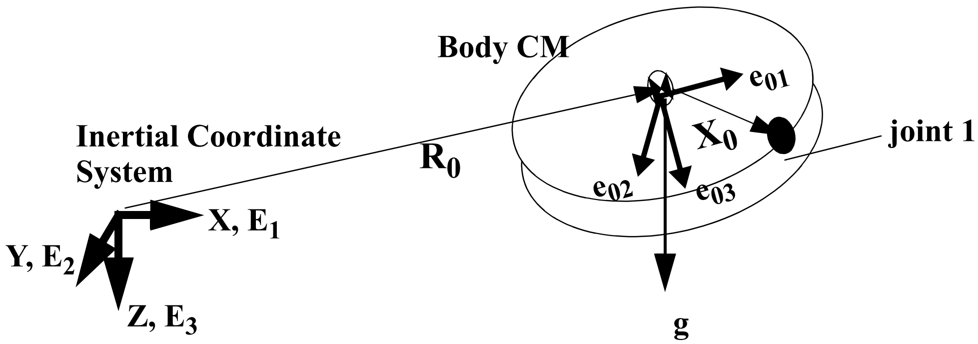

Figure 3. Inertial Coordinate System and Euler Angles

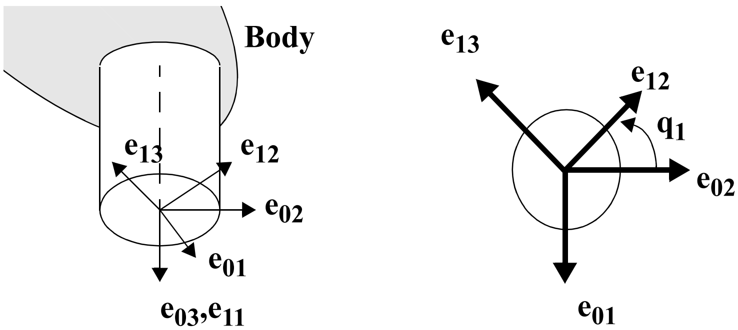

Figure 4. Definition of Shoulder Joint Angles (Link 1)

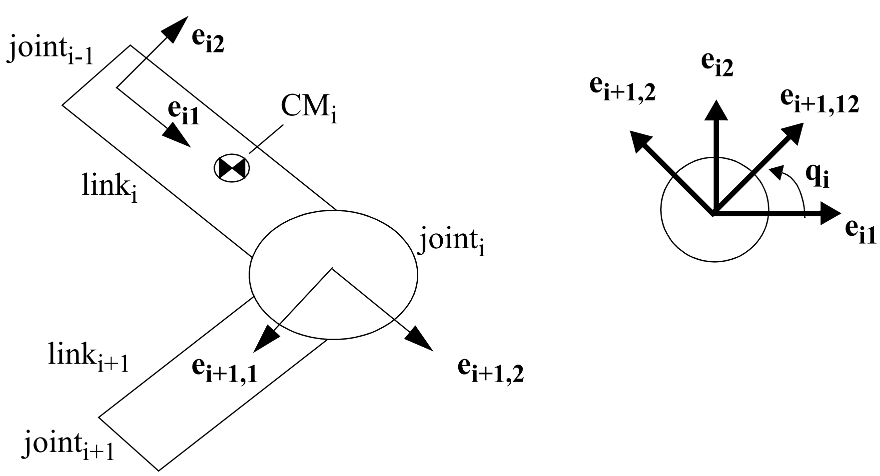

Figure 5. Definition of Hip, Knee, and Ankle Joint Angles (Links 2,3,4)

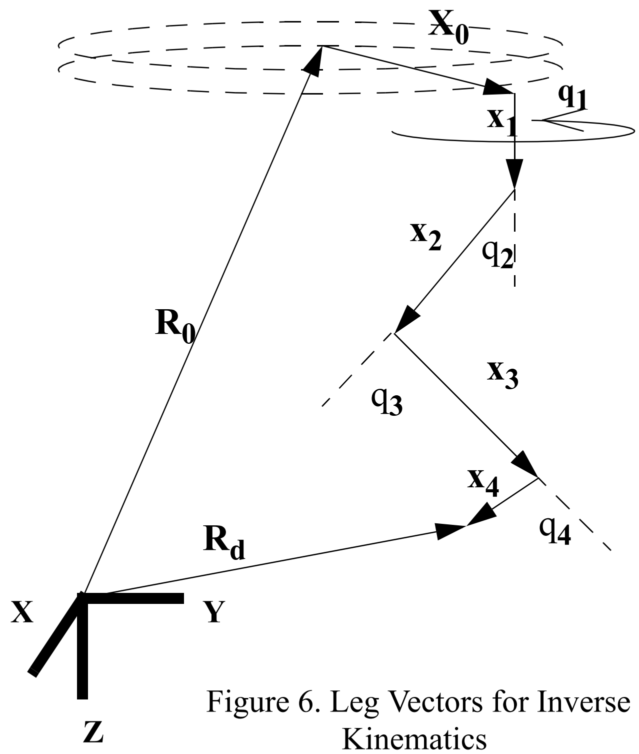

Figure 6. Leg Vectors for Inverse Kinematics